

GENERALIZATION OF MIRSKY'S THEOREM ON DIAGONALS AND EIGENVALUES OF MATRICES

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ABSTRACT. Mirsky proved that, for the existence of a complex matrix with given eigenvalues and diagonal entries, the obvious necessary condition is also sufficient. We generalize this theorem to matrices over any field and provide a short proof. Moreover, we show that there is a unique companion-matrix-type solution for this problem.

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If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of a complex matrix A of order n and d_1, \dots, d_n are its diagonal elements, then the sum of the λ_i 's is necessarily equal to the sum of the d_i 's. Mirsky [1] proved that the converse holds. If the data are real numbers, he proved that A can be chosen to be real as well. For a recent short proof of Mirsky's results see [2].

Instead of specifying the eigenvalues of a matrix we shall specify its characteristic polynomial. We shall work over any field F . Let

$$f(t) = t^n + c_{n-1}t^{n-1} + c_{n-2}t^{n-2} + \dots + c_0$$

be a monic polynomial over F .

Theorem 1.1. *Given a sequence d_1, \dots, d_n in F with $d_1 + \dots + d_n = -c_{n-1}$, there exists a unique sequence b_1, \dots, b_{n-1} in F such that $f(t)$ is the characteristic polynomial of the matrix*

$$A = [a_{ij}] = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 & b_1 \\ 1 & d_2 & 0 & & 0 & b_2 \\ 0 & 1 & d_3 & & 0 & b_3 \\ \vdots & & & & & \\ 0 & 0 & 0 & & d_{n-1} & b_{n-1} \\ 0 & 0 & 0 & & 1 & d_n \end{bmatrix}.$$

Proof. For any $S \subseteq \{1, \dots, n\}$, let A_S be the submatrix of A whose entries are the a_{ij} with $i, j \in S$, and let $M_S = \det A_S$. It suffices to prove that the system of $n-1$ equations

$$\sum_{S: |S|=n-k+1} M_S = (-1)^{n-k+1} c_{k-1}, \quad k = 1, \dots, n-1$$

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in $n - 1$ unknowns b_1, \dots, b_{n-1} has a unique solution in F . (By $|S|$ we denote the cardinality of S .) This is true because of the following claim: for each k we have

$$\sum_{S: |S|=n-k+1} M_S = (-1)^{n-k} b_k + g_k,$$

where g_k is a polynomial in the unknowns b_{k+1}, \dots, b_{n-1} only.

To prove this claim, it suffices to show that if b_k occurs in M_S and $|S| \leq n - k + 1$ then $S = \{k, \dots, n\}$. By the hypothesis b_k occurs in M_S , and so $\{k, n\} \subseteq S$ and there must exist a permutation π of S such that $\pi n = k$ and

$$\prod_{i \in S} a_{\pi i, i} \neq 0.$$

Hence, $\pi i \in \{i, i + 1\}$ for $i \in S \setminus \{n\}$. As $k \in S$ and $\pi n = k$, we have $\pi k = k + 1$. If $k < n - 1$ then $\pi(k + 1) \neq k + 1$ and so $\pi(k + 1) = k + 2$, etc. By repeating this argument, we conclude that $\{k, \dots, n\} \subseteq S$. As $|S| \leq n - k + 1$, it follows that $S = \{k, \dots, n\}$. This completes the proof of our claim and the theorem. \square

If $d_n = -c_{n-1}$ and all other $d_i = 0$, then A becomes the well known Frobenius companion matrix of $f(t)$:

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & & 0 & -c_1 \\ 0 & 1 & 0 & & 0 & -c_2 \\ \vdots & & & & & \\ 0 & 0 & 0 & & 0 & -c_{n-2} \\ 0 & 0 & 0 & & 1 & -c_{n-1} \end{bmatrix}.$$

The next theorem, which has much simpler proof, provides explicit formulae for the unknown elements b_1, \dots, b_{n-1} and establishes the existence assertion of Theorem 1.1, but not the uniqueness.

For $k = 0, 1, \dots, n - 1$, denote by $h_r(d_1, \dots, d_k)$ the sum of all monomials in d_1, \dots, d_k of degree r . (In particular $h_0(d_1, \dots, d_k) = 1$). Let I_n be the identity matrix of order n .

Theorem 1.2. *If $d_n = -c_{n-1} - d_1 - \dots - d_{n-1}$ and*

$$(1.1) \quad b_k = - \sum_{i=k-1}^n c_i h_{i-k+1}(d_1, \dots, d_k), \quad k = 1, \dots, n - 1,$$

where $c_n = 1$, then $\det(tI_n - A) = f(t)$.

Proof. Let $T = [t_{ij}]$ be the upper triangular matrix with entries $t_{ij} = h_{j-i}(d_1, \dots, d_i)$, $1 \leq i \leq j \leq n$. As all $t_{ii} = 1$, T is invertible. It suffices to verify that $AT = TC$, which is straightforward. \square

For example, if $n = 4$ then the above formulae read

$$\begin{aligned} b_1 &= -c_0 - c_1 d_1 - c_2 d_1^2 - c_3 d_1^3 - d_1^4, \\ b_2 &= -c_1 - c_2(d_1 + d_2) - c_3(d_1^2 + d_1 d_2 + d_2^2) - (d_1^3 + d_1^2 d_2 + d_1 d_2^2 + d_2^3), \\ b_3 &= -c_2 - c_3(d_1 + d_2 + d_3) - (d_1^2 + d_2^2 + d_3^2 + d_1 d_2 + d_1 d_3 + d_2 d_3). \end{aligned}$$

REFERENCES

- [1] L. Mirsky, Matrices with prescribed characteristic roots and diagonal elements, J. London Math. Soc. **33**, 14-21, 1958.
- [2] E.A. Carlen and E.H. Lieb, Short proofs of theorems of Mirsky and Horn on diagonals and eigenvalues of matrices, Electronic Journal of Linear Algebra **18**, 438-441, 2009.

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